The Fundamental Theorem of Calculus

We have now seen the two major branches of calculus:
1) differential (tangent line problem)
2) integral (area problem)

Leibniz and Newton independently discovered a connection between the two branches, stated informally, differentiation and (definite) integration are inverse operations.
THE FUNDAMENTAL THEOREM OF CALCULUS

(If \( f \) has an antiderivative \( F \) then you can find it this way....)

If a function \( f \) is continuous on a closed interval \([a, b]\) and \( F \) is an antiderivative of \( f \) on the interval \([a, b]\), then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

When applying the Fundamental Theorem of Calculus, follow the notation below:

\[
\int_{a}^{b} f(x) \, dx = F(b) \bigg|_{a}^{b} = F(b) - F(a)
\]
EX #1A:
\[
\int_{1}^{3} x^3 \, dx = \left[ \frac{x^4}{4} \right]_1^3 = \frac{(3)^4}{4} - \frac{(1)^4}{4} = \frac{81}{4} - \frac{1}{4} = \frac{20}{4}
\]

EX #1B:
\[
\int_{1}^{2} (x^2 - 3) \, dx = \left[ \frac{x^3}{3} - 3x \right]_1^2
\]
\[
= \left[ \frac{(2)^3}{3} - 3(2) \right] - \left[ \frac{(1)^3}{3} - 3(1) \right]
\]
\[
= \frac{8}{3} - 6 - \left( \frac{1}{3} - 3 \right)
\]
\[
= -10 - (-8)
\]
\[
= -\frac{2}{3}
\]

EX #1C:
\[
\int_{1}^{4} 3\sqrt{x} \, dx = 3 \int_{1}^{4} x^{\frac{1}{2}} \, dx = 3 \left[ \frac{2x^{\frac{3}{2}}}{3} \right]_1^4
\]
\[
= 3 \left[ \frac{2(4)^{\frac{3}{2}}}{3} - \frac{2(1)^{\frac{3}{2}}}{3} \right]
\]
\[
= \frac{16}{3} - \frac{2}{3}
\]
\[
= \frac{14}{3}
\]

EX #1D:
\[
\int_{0}^{\pi/4} \sec^2 x \, dx = \left[ \tan x \right]_0^{\pi/4}
\]
\[
= \tan \left( \frac{\pi}{4} \right) - \tan (0)
\]
\[
= 1 - 0
\]
\[
= \frac{1}{1}
\]
EX #2: A Definite Integral Involving Absolute Value

\[ \int_{0}^{2} |2x - 1| \, dx \]

Consider this:
1) zeros of \( f(x) \)
2) Definition of absolute value
3) Separate integral
4) Drawing:

(1) \( 2x - 1 = 0 \) \( \Rightarrow x = \frac{1}{2} \)

(4)

(2) \[ |2x - 1| = \begin{cases} 
-(2x-1) & \text{if } x < \frac{1}{2} \\
(2x-1) & \text{if } x \geq \frac{1}{2} 
\end{cases} \]

(3)

\[ \int_{0}^{\frac{1}{2}} -(2x-1) \, dx + \int_{\frac{1}{2}}^{2} (2x-1) \, dx \]

\[ = \left[ -x^2 + x \right]_{0}^{\frac{1}{2}} + \left[ x^2 - x \right]_{\frac{1}{2}}^{2} \]

\[ = \left( -\frac{1}{4} + \frac{1}{2} \right) - \left( 0 + 0 \right) + \left[ (4-2) - \left( \frac{1}{4} - \frac{2}{4} \right) \right] \]

\[ = \frac{1}{4} + \frac{8}{4} + \frac{1}{4} \]

\[ = \frac{5}{2} \]
EX #3: Finding Area

Find the area of the region bounded by the graph of $y = 3x^2 - 2x$, the x-axis, and the vertical lines $x = 1$ and $x = 4$.

1. Draw

2. Find $F(x)$

\[
\int_{1}^{4} (3x^2 - 2x) \, dx
\]

\[
\left[ x^3 - x^2 \right]_{1}^{4} = (4^3 - 4^2) - (1^3 - 1^2)
\]

\[
= (64 - 16) - 0
\]

\[
= 48
\]

Area = 48 $u^2$
THE MEAN VALUE THEOREM FOR INTEGRALS

The area of a region under a curve is greater than the area of an inscribed rectangle and less than the area of a circumscribed rectangle. The Mean Value Theorem for Integrals states that somewhere “between” the inscribed and circumscribed rectangles there is a rectangle whose area is precisely equal to the area of the region under the curve.

**Theorem:**

If $f$ is continuous on the closed interval $[a, b]$, then there exists a number $c$ in the closed interval $[a, b]$ such that

$$
\int_{a}^{b} f(x) \, dx = f(c)(b-a)
$$

**NOTE:** Mean Value Theorem doesn’t tell how to find “$c$”, merely guarantees existence of at least one number $c$ in the interval.

**To find $c$...**

1) find average value of $f(x)$
2) set $f(x)$ equal to average value
3) Solve for “$c$”
4) Determine value(s) that satisfy interval $[a,b]$. 
Average Value of a Function

The value of \( f(c) \) given in the Mean Value Theorem for Integrals is called the **average value** of \( f \) on the interval \([a, b]\).

**Definition of the Average Value of a Function on an Interval:**

If \( f \) is integrable on the closed interval \([a, b]\), then the average value of \( f \) on the interval is

\[
\frac{1}{b-a} \int_a^b f(x) \, dx
\]

EX #4: Find the average value of \( f(x) = 3x^2 - 2x \) on the interval \([1, 4]\)

From ex #3 we know \( A = 48 \)

\[
\frac{1}{4-1} \int_1^4 (3x^2 - 2x) \, dx = \frac{1}{3} \left[ x^3 - x^2 \right]_1^4
\]

\[
= \frac{1}{3} \left[ 48 \right] = 16
\]

The average value of \( f(x) \) over the interval from \([1, 4]\) is 16... So there is at least one value on the interval where a rectangle can be drawn that is 16 units tall and 3 units wide.

\((16)(3) = 48\)

(area under curve)

EX#4B: Find the value(s) of \( c \) guaranteed by the Mean Value Theorem for Integrals.

\[
\begin{align*}
\text{function} & \quad = \text{average value} \\
3x^2 - 2x & \quad = 16 \\
3c^2 - 2c & \quad = 16 \\
3c^2 - 2c - 16 & \quad = 0 \\
(3c-8)(c+2) & \quad = 0 \\
c & \quad = \frac{8}{3}, \quad c = -2
\end{align*}
\]
THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

(*Every function f that is continuous on an open interval, has an antiderivative F on the interval...*)

If $f$ is continuous on an open interval $I$ containing $a$, then, for every $x$ in the interval.

$$\frac{dF}{dt} = \frac{d}{dx}\left[\int_{a}^{x} f(t)\,dt\right] = f(x)$$

Similarly,

$$\frac{d}{dx}\left[\int_{a}^{u} f(t)\,dt\right] = \frac{du}{dx} \cdot f(u)$$

And yet another way to interpret the Second Fundamental Theorem:

If $f(x)$ is a continuous and differentiable function,

$$\frac{d}{dx}\left[\int_{a}^{f(x)} g(y)\,dy\right] = g(f(x)) \cdot f'(x)$$

Now, let’s explain all of this...

Say you’re taking the derivative of a definite integral whose lower bound is a constant and whose upper bound contains a variable.

If you take the derivative of the entire integral with respect to the variable in the upper bound, the answer will be the function inside the integral sign (unintegrated), with the upper bound plugged in, multiplied by the derivative of the upper bound. All you need to do is learn the pattern.
EX #5: For \( F(x) = \int_{2}^{x} \frac{1}{3t^2} \, dt \), find

a.) \( F(x) = \frac{1}{3} \int_{2}^{x} t^{-2} \, dt \)

\[ = \frac{1}{3} \left[ -x^{-1} \right]_{2}^{x} = \left[ -\frac{1}{3x} \right]_{2}^{x} = -\frac{1}{3x} + \frac{1}{6} \]

b.) \( F(2) = \int_{2}^{2} \frac{1}{3t^2} \, dt = 0 \)

c.) \( F'(x) = \frac{1}{3x^2} \Rightarrow F'(x) = -\frac{1}{3x} + \frac{1}{6} \)

d.) \( F'(2) = \frac{1}{3(2)^2} = \frac{1}{12} \)
EX #6: Evaluate $\frac{d}{dx} \int_a^x \left( \sqrt{1+t^2} \right) dt$

\[ \sqrt{1+x^2} \]

EX #7: Find $F'(x)$; $F(x) = \int_{-\pi/3}^{x^3} \cos(t) \, dt$

Method #1

$F(x) = \left[ \sin x \right]_{-\pi/3}^{x^3}$

$F(x) = \sin x^3 - \sin \left( -\frac{\pi}{3} \right)$

$F(x) = \sin x^3 + \frac{\sqrt{3}}{2}$

$F'(x) = \cos x^3 (3x^2)$

Method #2

$F'(x) = \frac{dF}{dx} \frac{du}{dx}$

$\frac{d}{dx} \left[ \int_{-\pi/3}^{x^3} \cos t \, dt \right] \frac{du}{dx}$

$F'(x) = \cos u \left( 3x^2 \right)$

$F'(x) = \cos x^3 (3x^2)$