DIFFERENTIAL EQUATIONS: GROWTH AND DECAY

In order to solve a more general type of differential equation, we will look at a method known as *separation of variables*. The general strategy is to rewrite the equation so that each variable occurs on only one side of the equation.

EX #1: Solve the differential equation $y' = \dfrac{3x}{y}$

$$\frac{dy}{dx} = \frac{3x}{y}$$

$$y\,dy = 3x\,dx$$

$$\int y\,dy = \int 3x\,dx$$

$$\frac{1}{2}y^2 = \frac{3}{2}x^2 + C$$

$$y^2 = 3x^2 + C$$

When integrating both sides, it is not necessary to add constants to both sides. Results will be the same.

EX #2: Solve the differential equation $\dfrac{dy}{dx} = 2 - y$

$$\frac{dy}{dx} = 2-y$$

Separate variables and differentials

$$\frac{dy}{2-y} = dx$$

$$\int \frac{dy}{2-y} = \int dx$$

$$\int \frac{1}{2-y}\,dy = \int dx \quad \Leftarrow$$

Let $u = 2-y$

$du = -dy$

$-du = dy$

$-\int \frac{1}{u}\,du$

$-\ln|u| + C$

$$\underline{\underline{-\ln|2-y| = x + C}}$$

or

$$\underline{x = C - \ln|2-y|}$$

EXPONENTIAL GROWTH AND DECAY MODELS

If y is a differentiable function of t such that $y > 0$ and $y' = ky$ for some constant k, then

$$y = Ce^{kt}$$

C is the **initial value** of y, and k is the **proportionality constant. Exponential growth** occurs when $k > 0$, and **exponential decay** occurs when $k < 0$.

EX #3: A slow economy caused a company's annual revenues to drop from $530, 000 in 2008 to $386,000 in 2010. If the revenue is following an exponential pattern of decline, what is the expected revenue in 2012?

model: $y = Ce^{kt}$

we know
(0, 530,000)
(2, 386,000)

$t=0$, 2008
$t=2$, 2010
$t=4$, 2012

C = initial amt = 530000. We can use given data to find k-value.

$$386{,}000 = 530{,}000e^{2k}$$

$$\frac{386}{530} = e^{2k}$$

$$\ln\left(\frac{386}{530}\right) = \ln e^{2k}$$

$$\ln\left(\frac{386}{530}\right) = 2k \cancel{\ln e}$$

$$\frac{1}{2}\ln\left(\frac{386}{530}\right) = k$$

$$k \approx -0.15852$$

model: $y = 530{,}000\, e^{-0.15852t}$

when $t = 4$

$$y \approx 530000e^{(-.15852)(4)}$$

$$y \approx 530{,}000\, e^{(-0.63408)}$$

$$\boxed{y \approx \$281{,}124.32}$$

expected revenue for 2012

EX #4: Suppose a population of insects increases according to the law of exponential growth. There were 130 insects after the third day of the experiment and 380 insects after the seventh day. Approximately how many insects were in the original population?

$y = Ce^{kt}$

$(3, 130)$
$(7, 380)$
$t =$ in days

Write two equations with given data:

$130 = Ce^{3k}$ eq.#1
$380 = Ce^{7k}$ eq#2

Solve eq.#1 for C, then substitute into eq#2 to find k-value.

$130 = Ce^{3k}$
$C = \dfrac{130}{e^{3k}}$
$\underline{C = 130e^{-3k}}$

$380 = Ce^{7k}$
$380 = (130e^{-3k})e^{7k}$
$380 = 130e^{4k}$
$\dfrac{38\cancel{0}}{13\cancel{0}} = e^{4k}$
$\ln\left(\dfrac{38}{13}\right) = 4k \cancel{\ln e}$
$\dfrac{1}{4}\ln\left(\dfrac{38}{13}\right) = k$
$k \approx 0.268159$

Growth model:
$y = Ce^{0.268159t}$

when $t = 3$

$130 \approx Ce^{(0.268159)(3)}$
$130 \approx Ce^{.80448}$
$\dfrac{130}{e^{.80448}} \approx C$
$\underline{C \approx 58 \text{ insects, initially}}$

EX #5: A zircon sample contains 4000 atoms of the radioactive element 235U. Given that 235U has a half- life of 700 million years, how long would it take to decay to 125 atoms?

t = in millions of years

model: $y = 4000\left(\frac{1}{2}\right)^{t/700}$

$125 = 4000\left(\frac{1}{2}\right)^{t/700}$

t = 0, 4000 atoms
t = ?, 125 atoms

$\frac{125}{4000} = \frac{1}{2}^{t/700}$

$\ln\left(\frac{125}{4000}\right) = \ln\left(\frac{1}{2}\right)^{t/700}$

$\ln\left(\frac{125}{4000}\right) = \frac{t}{700}\ln\left(\frac{1}{2}\right)$

$\frac{\ln\left(\frac{125}{400}\right)}{\ln(2)} = \frac{t}{700}$

$5 = \frac{t}{700}$

$\boxed{t = 3500}$ million years before zircon sample degrades to 125 atoms

EX #6: The number of bacteria in a culture is growing at a rate of $3000e^{2t/5}$ per unit of time *t*.
At $t = 0$, the number of bacteria present was 7,500. Find the number present at $t = 5$.

$$\frac{dB}{dt} = 3000e^{2t/5}$$

$$\int dB = \int 3000e^{2t/5}\,dt$$

$$B = 3000e^{2t/5} \cdot \frac{5}{2}$$

$$B = 7500e^{2t/5}$$

$$B(t) = 7500e^{2t/5}$$

$$B(5) = 7500e^{2}$$

$$B(5) = 55417.92$$

$$B(5) = 55418$$